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EE 364 - Fall 2026

# Week 1
# Supplemental Notes



## Logical operations

Compare:
Knowledge: justified true belief.

Q: what is truth?

A: A function, $t: \{\text{statements}\} \longrightarrow \{0, 1\}$.

e.g. P and Q are statements with truth value $\to$ T, F

$t(P \,\&\, Q) = \min(t(P), t(Q))$

| P | Q | P & Q |
|---|---|---|
| 1 | 1 | 1 |
| 0 | 1 | 0 |
| 1 | 0 | 0 |
| 0 | 0 | 0 |

$t(P \vee Q) = \max(t(P), t(Q))$

| P | Q | P ∨ Q |
|---|---|---|
| 1 | 1 | 1 |
| 0 | 1 | 1 |
| 1 | 0 | 1 |
| 0 | 0 | 0 |

$t(\sim P) = 1 - t(P)$

| P | ~P |
|---|---|
| 1 | 0 |
| 0 | 1 |

"OR" $\min(1, \cdot)$

$t(P \rightarrow Q) = \min(1, \underbrace{1 - t(P)}_{t(\sim P)} + \underbrace{t(Q)}_{t(Q)})$

| P | Q | P → Q |
|---|---|---|
| 1 | 1 | 1 |
| 0 | 1 | 1 |
| 1 | 0 | 0 |
| 0 | 0 | 1 |

$t(P \oplus Q) = |t(P) - t(Q)|$

| P | Q | P ⊕ Q |
|---|---|---|
| 1 | 1 | 0 |
| 0 | 1 | 1 |
| 1 | 0 | 1 |
| 0 | 0 | 0 |

$t(P \leftrightarrow Q) = 1 - |t(P) - t(Q)|$

| P | Q | P ↔ Q |
|---|---|---|
| 1 | 1 | 1 |
| 0 | 1 | 0 |
| 1 | 0 | 0 |
| 0 | 0 | 1 |

# Intuition / Implication.

Q: why [false $\rightarrow$ true] is true.

A: by definition. But consider

"trivially true":
(i.e. true for all inputs)

If given number x less than 10 then x less than 100

[i.e. $x < 10 \rightarrow x < 100$]

Case 1: $x = 5$, $5 < 10 \rightarrow 5 < 100$ true $\rightarrow$ true

true

Case 4: $x = 200$, $200 < 10 \rightarrow 200 < 100$ false $\rightarrow$ false

true

Case 2: $x = 50$, $50 < 10 \rightarrow 50 < 100$ false $\rightarrow$ true

true

Case 3: no such x, $x < 10$ and $x \not< 100$.

$\therefore$ true for all inputs.

Theorem (1) $P \& Q \implies P$

(2) $P \implies P \vee Q$

Prf: by (binary) truth table.

| $P$ | $Q$ | $P$ | $\&$ | $Q$ | $\implies$ | $P$ | $P$ | $\implies$ | $P$ | $\vee$ | $Q$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 0 | 1 | 0 | 0 | 1 | 1 | 0 | 0 | 1 | 0 | 1 | 1 |
| 1 | 0 | 1 | 0 | 0 | 1 | 1 | 1 | 1 | 1 | 1 | 0 |
| 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 1 | 0 | 0 | 0 |

QED.

tautology
"equivalent to truth"

$\therefore$ valid since statement is equivalent to "1" for all inputs

$\exists \longleftrightarrow$ "Some"
$\forall \longleftrightarrow$ "All"

Arbitrary $A_\alpha$, AND $\&, \cap, \forall$.

OR $\vee, \cup, \exists$

$$\bigcap_\alpha A_\alpha \overset{D.M}{=} \left(\bigcup_\alpha A_\alpha\right)^c$$

$$x \in \bigcap_\alpha A_\alpha \longrightarrow \forall \alpha, x \in A_\alpha$$

$$\bigcup_\alpha A_\alpha \overset{D.M.}{=} \left(\bigcap_\alpha A_\alpha\right)^c$$

$$x \in \bigcup_\alpha A_\alpha \longrightarrow \exists \alpha' : x \in A_{\alpha'}$$

Lemma: little theorem used to prove big theorem

Corollary: special case of theorem

# Models

**Defn:** Models are abstractions of reality with simplified behavior and cause-effect

**Note:** A model is only as good as the modeller and the assumptions that underlie the model.

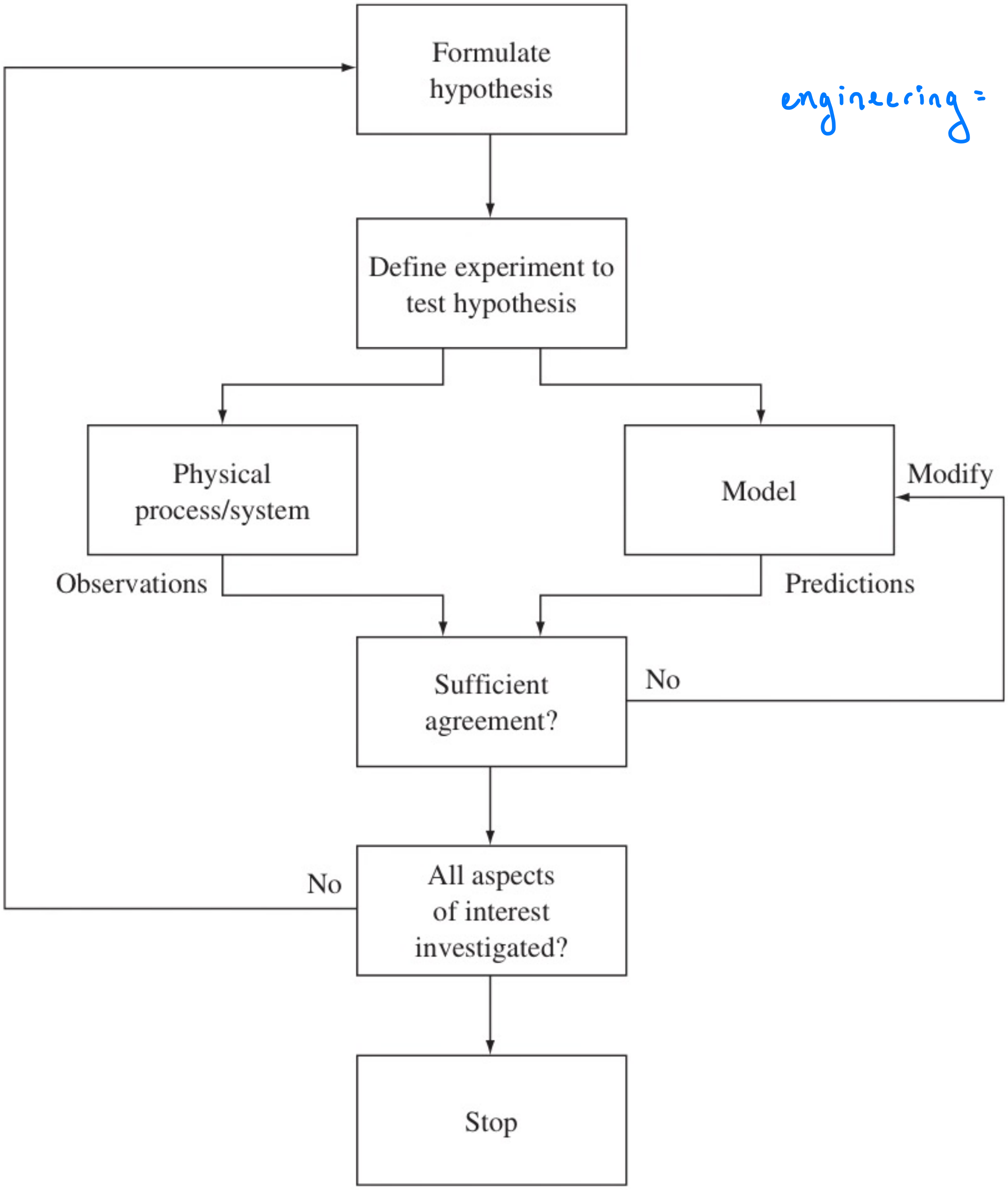
A <u>useful</u> model explains all relevant aspects → i.e. what you care about

**Ex:** Assuming "bell shaped" model for class grades.
(i.e., "the curve")

**Goal:** explain observed behavior.

Ideal: simple and understandable rules.

engineering = science + money




Each step requires modeller input and expert opinion.

Iterative process continues until model is sufficient
(or run out of time/money)
"good enough"

# Types of models

- Deterministic. Same input produce same output

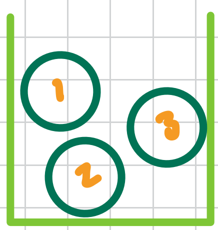
  Ex: $V = iR$.

- Random Same input may produce different output.

Defn: A random experiment is an observation that varies in an "unpredictable" way when repeated under the same conditions

"Näive" probability theory.

Ex: 3-balls in urn experiment




1. Shake and randomly select ("random sample")
2. Record #. Return ball ("with replacement")

$\therefore$ Sample space $\Omega = \{1, 2, 3\}$.

Repeat experiment

Outcome varies unpredictably. Cannot infer single outcome given history

# Statistical regularity

Regularity is a means to quantify predictability in the long run.

Ex: Weak law of large numbers (WLLN)

each called a "trial"

Average of outcomes from a long sequence of experiments yield approximately the same value

Ex: Central limit theorem (CLT)

Standardized average from a long sequence of experiments follow approximately the same "distribution"

Ex: 3-balls in urn.

Define $N_1(n)$, $N_2(n)$, $N_3(n)$ as number of times you observe 1, 2, 3 after n-flips.

$N_1(n)$, $N_2(n)$, $N_3(n)$ are random.

Define the relative frequency

$$f_k(n) = \frac{N_k(n)}{n} \quad \text{for } n = 1, 2, 3$$

After you run the experiment and compute $N_1(n)$, $N_2(n)$, $N_3(n)$ they are not random. "Realization" of random variable

Statistical regularity means $f_k(n)$ varies less as n "gets large"

specifically: $\lim_{n \to \infty} f_k(n) = p_k$

constant. called the probability of outcome k.

## Problems with this "frequentist" approach:

1. Problems with the limit (in a mathematical sense)

   must $\lim f_n(n)$ always converge? No.

2. Impossible to repeat experiment infinite times

   only finite samples. How to compute $p_k$?

3. How to operate for experiments you cannot repeat?

   e.g., election outcome

4. How to handle experiments with a continuous sample space. (continuum of outcomes)

   e.g., pick random number in interval $[0,1]$.

$\therefore$ Need a more robust approach but maintains consistency with frequentist approach / intuitive understand of probability.

# Axiomatic Probability Theory.

Idea: in random experiments define outcomes and occurance of events as sets

General idea: Probability is about measuring relative size of sets.

Basic structure:

Suppose and experiment yields a random outcome

(1) The set of all possible results $\Omega$ is the "sample space"

(2) Elements of the sample space are "outcomes"

(3) Events are a special class of subsets of the sample space called "measurable sets"

(4) The probability of an event is the "size" of the set relative to the whole sample space ($\Omega$)

Axiomatic probability theory uses the language of sets.

# Language of sets

$$A \cap B = \{x \in X : x \in A \ \& \ x \in B\}$$

- set intersection

$$A \cup B = \{x \in X : x \in A \vee x \in B\}$$

- set union

$$A^c : \{x \in X : x \notin A\}$$

- set complement

$$A \subset B \longleftrightarrow \forall x : x \in A \longrightarrow x \in B.$$

- set subset

$$A - B = A \cap B^c$$

- set difference

result is a proposition (not set)

$$A = B \longleftrightarrow A \subset B \ \& \ B \subset A.$$

- set equality

$$A \triangle B = (A \cap B^c) \cup (B \cap A^c)$$

- symmetric set difference

# De Morgans

$$A \cap B = (A^c \cup B^c)^c$$

$$A \cup B = (A^c \cap B^c)^c$$

$$\bigcap_\alpha A_\alpha = \left(\bigcup_\alpha A_\alpha^c\right)^c$$

$$\bigcup_\alpha A_\alpha = \left(\bigcap_\alpha A_\alpha^c\right)^c$$

compare: $\min(x,y) = -\max(-x,-y)$
$\max(x,y) = -\min(-x,-y)$

Logical diversion – "how to proof"

* if-then  "if $p$ then $q$", "$q$ if $p$", "$p \to q$"

  1. Suppose "if" part is true
  2. Show "then" part is true ("prove") or false (disprove)

* if and only if  "$p$ if and only if $q$", "$p$ iff $q$", "$p \leftrightarrow q$"

  1. Prove $p \to q$
  2. Prove $q \to p$

  must do <u>both</u>

* subset  "$X \subset Y$"

  1. Choose arbitrary element $x \in X$.
  2. Show $x \in Y$.

* equality  "$X = Y$"

  1. Show $X \subset Y$
  2. Show $Y \subset X$

  must do <u>both</u>

Note: Venn diagrams are <u>NOT</u> proofs. helpful tools use as guidance.

Ex: $A \cap B \subset A \subset A \cup B$.

Prf: Claim 1: $A \cap B \subset A$

$x \in A \cap B$ assump.

$\therefore$ $x \in A$ AND $x \in B$ defn $\cap$

$\therefore$ $x \in A$ Lemma (TT above)

$\therefore$ $A \cap B \subset A$ Defn $\subset$ QED.

Claim 2: $A \subset A \cup B$.

$x \in A$ assump.

$\therefore$ $x \in A$ OR $x \in B$ Lemma (TT above)

$\therefore$ $x \in A \cup B$ Defn $\cup$

$\therefore$ $A \subset A \cup B$ Defn $\subset$

Q: What is probability

A: It is a function. $P: \mathcal{Q} \longrightarrow [0,1]$.

(2) CAT

- definition of a probability measure

(all possible outcomes)

sample space

all possible events.

Defn: Suppose $(X, \mathcal{Q})$ is a measurable space. Then $P: \mathcal{Q} \longrightarrow [0,1]$ is a probability measure iff $P$ is CAT.

CA: Countable Additive

$$P\left(\bigcup_{k=1}^{\infty} A_k\right) = \sum_{k=1}^{\infty} P(A_k) \quad \text{if } A_i \cap A_j = \phi \text{ when } i \neq j$$

($A_i$ and $A_j$ are "mutually exclusive")

T: Total set $X$ gets unit mass:

$$P(X) = 1$$

Defn: $(X, \mathcal{Q}, P)$ is a probability space iff $(X, \mathcal{Q})$ is a measurable space and $P: \mathcal{Q} \longrightarrow [0,1]$ is CAT.

Ex: Prove or disprove $[A \cup B = B] \longrightarrow [A \subset B]$.

Prf: method of proof for " $\longrightarrow$ "

1. Assume LHS
2. Determine RHS true/false.

Suppose $A \cup B = B$. $[\therefore$ show $A \subset B]$.

| P | Q | $P \to Q$ |
|---|---|---|
| 1 | 1 | 1 |
| 0 | 1 | 1 |
| 1 | 0 | 0 |
| 0 | 0 | 1 |

only row to worry about!

1. pick $x \in A$ — Assump.
2. $\therefore x \in A$ OR $x \in B$ — Defn $\vee$
3. $\therefore x \in A \cup B$ — Defn $\cup$ ← like LHS, that's the key
4. $\therefore x \in B$ — Since $A \cup B = B$ (by assump.)
5. $\therefore A \subset B$ — Defn $\subset$

Ex: $P[\emptyset] = 0$. → CAT, $\therefore$ only know $P[X] = 1$.

Prf: $X = X \cup \emptyset$.

X and $\emptyset$ mutually exclusive since $X \cap \emptyset = \emptyset$. and $X \cup \emptyset = X$.

$$\therefore P[X] = P[X \cup \emptyset] \overset{C.A.}{=} P[X] + P[\emptyset]$$

$$\therefore \overset{T}{1} = 1 + P[\emptyset]$$

$$\therefore P[\emptyset] = 0.$$

**Ex:** Monotony $\quad A \subset B \longrightarrow P(A) \leq P(B)$

**Prf:** Suppose $A \subset B$

$$B = B \cap X = B \cap (A \cap A^c)$$

$$= (B \cap A) \cap (B \cap A^c) \qquad \text{Distributivity (proof req'd)}$$

$$= A \cap (B \cap A^c) \qquad A \subset B \text{ (by hypo)}$$

$$\therefore P[B] = P[A] + P[B \cap A^c] \qquad (A) \cap (B \cap A^c) = \emptyset. \;\therefore \text{mut. exclusive + CA}$$

$$\therefore \quad \geq P[A]$$

**Ex:** Addition Theorem $\quad P(A) + P(B) = P(A \cup B) - P(A \cap B)$

$$A = A \cap X = A \cap (B \cup B^c)$$

$$B = B \cap X = B \cap (A \cup A^c)$$

$$\therefore \; P(A) + P(B) = P(\underbrace{A \cap (B \cup B^c)}_{(A\cap B) \cup (A \cap B^c)}) + P(\underbrace{B \cap (A \cup A^c)}_{(B \cap A) \cup (B \cap A^c)})$$

disjoint $\qquad$ disjoint

$$= P(A \cap B) + \underbrace{P(A \cap B^c) + P(A \cap B) + P(A^c \cap B)}_{= P(A \cup B) \text{ next page}}$$

$$= P(A \cap B) + P(A \cup B)$$

Ex: Addition Theorem (detailed proof)

$$P(A) + P(B) = P(A \cup B) + P(A \cap B)$$

Claim: $A \cup B = (A-B) \cup (A \cap B) \cup (B-A)$

$$= \underbrace{(A \cap B^c)}_{\substack{\text{prove} \\ P[A] - P[A \cap B]}} \cup (A \cap B) \cup \underbrace{(B \cap A^c)}_{\substack{\text{prove} \\ P[B] - P[A \cap B]}}$$

Lemma: $A \cup B = (A-B) \cup (A \cap B) \cup (B-A)$

Claim 1: $A \cup B \subset (A-B) \cup (A \cap B) \cup (B-A)$

| | | |
|---|---|---|
| 1. | pick $x \in A \cup B$ | Ass. |
| 2. | $\therefore x \in A$ OR $x \in B$ | defn $\cup$. |

Case 1: $x \in A$ AND $x \notin B$.

| | |
|---|---|
| $\therefore x \in A$ AND $x \in B^c$ | defn c |
| $\therefore x \in A \cap B^c$ | defn $\cap$ |
| $\therefore x \in A - B$ | defn $-$ |

Case 2: $x \notin A$ AND $x \in B$

| | |
|---|---|
| $\therefore x \in A^c$ AND $x \in B$. | defn c |
| $\therefore x \in B \cap A^c$ | defn $\cap$ |
| $\therefore x \in B - A$ | defn $-$ |

Case 3: $x \in A$ AND $x \in B$.

| | |
|---|---|
| $\therefore x \in A \cap B$. | defn $\cap$. |

But how to justify using cases? propositional logic, next page.

3. $\therefore\ x \in (A-B) \cup (A \cap B) \cup (B-A)$ lemma

4. $\therefore\ A \cup B \subset (A-B) \cup (A \cap B) \cup (B-A)$ defn $\subset$

QED, claim 1.

Lemma: $P \vee Q \longleftrightarrow (P \,\&\, \sim Q) \vee (P \,\&\, Q) \vee (\sim P \,\&\, Q)$

| P | Q | $P \vee Q$ | $\longrightarrow$ | $(P \,\&\, \sim Q)$ | $\vee$ | $(P \,\&\, Q)$ | $\vee$ | $(\sim P \,\&\, Q)$ |
|---|---|---|---|---|---|---|---|---|
| 1 | 1 | 1 1 1 | 1 | 1 0 0 | | 1 1 1 | 1 | 0 0 1 |
| 0 | 1 | 0 1 1 | 1 | 0 0 0 | | 0 0 1 | 1 | 1 1 1 |
| 1 | 0 | 1 1 0 | 1 | 1 1 1 | | 1 0 0 | 1 | 0 0 0 |
| 0 | 0 | 0 0 0 | 1 | 0 0 1 | | 0 0 0 | 0 | 1 0 0 |

$\therefore$ valid $\therefore$ QED. (Lemma)

Claim 2: $(A-B) \cup (A \cap B) \cup (B-A) \subset A \cup B$

1. pick $x \in (A-B) \cup (A \cap B) \cup (B-A)$. Ass.
2. $\therefore\ x \in (A-B)$ OR $x \in A \cap B$ OR $x \in B-A$ defn $\cup$

Case 1: $x \in A-B$.

1. $\therefore\ x \in A \cap B^c$ defn $-$
2. $\therefore\ x \in A$ AND $x \in B^c$ defn $\cap$
3. $\therefore\ x \in A$ Lemma: $P \,\&\, Q \longrightarrow P$.
4. $\therefore\ x \in A$ OR $x \in B$ Lemma: $P \longrightarrow P \vee Q$.
5. $\therefore\ x \in A \cup B$ defn $\cup$

Case 2 : $x \in A \cap B$ $\checkmark$

Case 3 : $x \in B - A$ $\checkmark$

3. $\therefore$ $x \in A \cup B$.

4. $\therefore$ $(A-B) \cup (A \cap B) \cup (B-A) \subset A \cup B$. defn $\subset$

QED, Claim 2.

$\therefore$ $A \cup B = (A-B) \cup (A \cap B) \cup (B-A)$ defn $=$, Claim 1 & 2.

QED, Lemma.

$$\therefore \; P(A) + P(B) = P(A-B) + P(A \cap B)$$
$$+ P(B-A) + P(A \cap B)$$
$$= P\big((A-B) + P(A \cap B) + P(B-A)\big) + P(A \cap B)$$
disjoint, CA.
$$= P(A \cup B) + P(A \cap B)$$
Lemma.

QED

quod erat demonstrandum
(which was to be demonstrated)